11/8/25 "Localization of KM-reps" (Speaker: Kevin Lin; notes: Nikolay)

BB localization: $\mathfrak{g}\text{-Mod} \underset{\Gamma}{\overset{Loc}{\rightleftarrows}} D\text{-mod}(X)$, $X = G/B$

$$\mathrm{Maps}(S, X_{dR}) = \mathrm{Maps}(S^{red}, X), \quad X = \text{finite-type scheme.}$$

$$D\text{-mod}(X) := \mathrm{ICoh}(X_{dR}) = \text{ind-coherent sheaves on } X_{dR}$$

$X \xrightarrow{\pi} X_{dR}$.

$$\begin{tikzcd} \mathrm{ICoh}(X_{dR}) \arrow[r, phantom, "\simeq"] \arrow[d, shift left, "\pi^!"] \arrow[dr, "oblv"'] & D(X) \\ \mathrm{ICoh}(X) \arrow[u, shift left, "\pi_*"] \arrow[ur, "ind"'] & \end{tikzcd}$$

$$\mathfrak{g}\text{-mod}(X) := \mathrm{ICoh}(B\exp\mathfrak{g} = \bullet/\exp\mathfrak{g})$$

$$\begin{tikzcd} pt \arrow[d, "i"] & \mathrm{Vect} \arrow[d, shift right, "i_*"'] & \mathrm{QCoh}(pt) \\ B\exp(\mathfrak{g}) & \mathrm{ICoh}(B\exp\mathfrak{g}) \arrow[u, shift right, "i^!"'] & \mathrm{QCoh}(B\exp(\mathfrak{g})) \arrow[u, "i^*"] \end{tikzcd}$$

$$\begin{tikzcd} & \exp(\mathfrak{g})\backslash X \arrow[dl, "q"'] \arrow[dr, "p"] & \\ \exp(\mathfrak{g})\backslash pt & & X_{dR} \end{tikzcd}$$

Then $\Gamma = q_* p^!$

$Loc = p_* q^*$

Everything has an action of $D(G) = \mathrm{ICoh}(G_{dR})$

$G_{dR}/G = \exp(\mathfrak{g})\backslash pt$ since $\exp(\mathfrak{g}) \to G \to G_{dR}$

$\to G_{dR} = G/\exp(\mathfrak{g})$

$$\begin{tikzcd} & G_{dR} \times^G X \arrow[dl] \arrow[dr] & \\ G_{dR}/G & & X_{dR} \end{tikzcd}$$

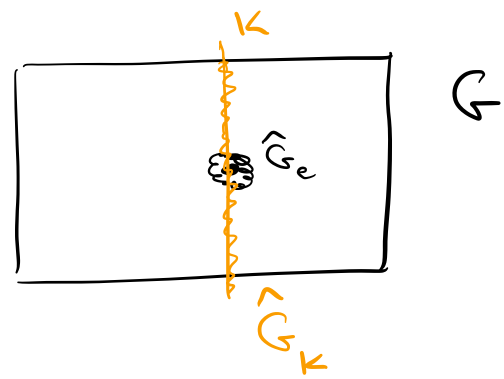
For any subgroup $K \subseteq G$, can take K-invariants

$D(K)$-invariants

$$\mathfrak{g}\text{-mod}^K \underset{\Gamma}{\overset{\mathrm{Loc}}{\rightleftarrows}} D(X)^K = D(K \backslash X)$$

$\mathfrak{g}\text{-mod}^K \overset{\text{Old}}{=}$ HC modules $=$ reps of $\mathfrak{g}$ + integral of k-action to K

$$\overset{\text{Modern}}{=} \mathrm{ICoh}\left(K_{dR} \backslash B\exp\mathfrak{g}\right) = \mathrm{ICoh}\left(B\hat{G}_K\right)$$



Affine case: Expect

$$\mathcal{L}\mathfrak{g}\text{-mod}^{\mathcal{L}^+G} \underset{\Gamma}{\overset{\mathrm{Loc}}{\rightleftarrows}} D\text{-mod}(\mathrm{Bun}_G)$$

$$\mathcal{L}\mathfrak{g}\text{-mod} \underset{\Gamma}{\overset{\mathrm{Loc}}{\rightleftarrows}} D\text{-mod}(\mathrm{Bun}_G^{\infty x})$$

$$\begin{tikzcd}
\mathrm{Bun}_G^{\infty x} \arrow[r] \arrow[d] & \mathrm{Bun}_G \arrow[r] \arrow[d] & \mathrm{Bun}_G(X^\circ) \arrow[d] \\
\mathrm{pt} \arrow[r] & \mathrm{Bun}_G(D) \arrow[r] \arrow[d, phantom, "\wr"] & \mathrm{Bun}_G(D^\circ) \arrow[d, phantom, "\wr"] \\
& B\mathcal{L}^+(G) \arrow[r] & B\mathcal{L}G
\end{tikzcd}$$

trivial bundle $=$ pt

Lets describe $\mathcal{L}\mathfrak{g}\text{-mod}^{\mathcal{L}^+G}$

$\mathfrak{g}\text{-mod}^K = K$-reps + extension of k-action to $\mathfrak{g}$.

$K \longrightarrow G^\wedge_K \rightsquigarrow BK \longrightarrow BG^\wedge_K$, so take

$\mathrm{ICoh}\, BK$ + <u>descent</u>:

$$\begin{tikzcd} \cdots \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & BK \times_{BG^\wedge_K} BK \arrow[r, shift left] \arrow[r, shift right] & BK \arrow[r] & BG^\wedge_K \end{tikzcd}$$

$$BK \times_{BG^\wedge_K} BK \cong K\backslash G^\wedge_K / K$$

$$\begin{tikzcd} K\backslash G^\wedge_K/K \arrow[r] & K\backslash pt \\ G^\wedge_K/K \arrow[r] \arrow[u] & pt \arrow[u] \end{tikzcd}$$

$$\mathfrak{L}\mathfrak{g}\text{-mod}^{\mathfrak{L}^+G} = \lim\left(\mathrm{ICoh}\, B\mathfrak{L}^+G \rightrightarrows \mathrm{ICoh}(\mathfrak{L}^+G\backslash \mathfrak{L}G^\wedge_{\mathfrak{L}^+G}/\mathfrak{L}^+G) \substack{\to\\\to\\\to} \cdots\right)$$

$$\mathfrak{L}^+G\backslash \mathfrak{L}G/\mathfrak{L}^+G = \mathrm{Bun}_G(D) \times_{\mathrm{Bun}_G(D^\circ)} \mathrm{Bun}_G(D)$$

$$\mathfrak{L}^+G\backslash \mathfrak{L}G^\wedge_{\mathfrak{L}^+G}/\mathfrak{L}^+G = \left(\mathrm{Bun}_G D \times_{\mathrm{Bun}_G D^\circ} \mathrm{Bun}_G D\right)^\wedge_{\mathrm{Bun}_G(D)} = \mathrm{Hecke}_G^{loc,\wedge}.$$

$$\begin{tikzcd} \mathrm{Hecke}^{glob,\wedge}_{G,x} \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \mathrm{Bun}_G \arrow[r] \arrow[d, "\pi"] & \mathrm{Bun}_G(X^\circ) \arrow[d] \\ \mathrm{Hecke}^{loc,\wedge}_G \arrow[r, shift left] \arrow[r, shift right] & \mathrm{Bun}_G(D) \arrow[r] & \mathrm{Bun}_G(D^\circ) \end{tikzcd}$$

<u>Nick's thesis</u>: D-mod(Bun_G) same as $\mathrm{ICoh}(\mathrm{Bun}_G)$ plus equivariance for $\mathrm{Hecke}^{glob,\wedge}_{G,\mathrm{Ran}}$.

$$\begin{tikzcd} D(\mathrm{Bun}_G) \arrow[r, "\Gamma"] \arrow[d, "\mathrm{oblv}"'] & \mathfrak{L}\mathfrak{g}\text{-mod} \arrow[d, "\mathrm{oblv}"] \\ \mathrm{ICoh}(\mathrm{Bun}_G) \arrow[r, "\Gamma"] & \mathrm{Rep}(\mathfrak{L}^+G) \end{tikzcd}$$

"$\Gamma = \pi_*$"

Γ = discontinuous

$$\begin{tikzcd} D(\mathrm{Bun}_G) & \mathcal{L}\mathfrak{g}\text{-mod}^{\mathcal{L}^+G} \arrow[l, "\mathrm{loc}"'] \\ \mathrm{ICoh}(\mathrm{Bun}_G) \arrow[u, "\mathrm{ind}"] & \mathrm{Rep}(\mathcal{L}^+G) \arrow[l, "\mathrm{loc}"] \arrow[u, "\mathrm{ind}"'] \end{tikzcd}$$

Example $\quad \mathbb{V} = \text{vacuum} = \mathrm{ind}(\underset{k}{\underline{\mathrm{triv}}})$

$\mathrm{Loc}\,\mathbb{V} = D_{\mathrm{Bun}_G}$

Want Γ to send compact to compact.
But D_{Bun_G} not quasi-compact

Let $\mathcal{U} \subseteq \mathrm{Bun}_G$ be q.cpt open subset, replace all "Bun_G" in diagrams w/ "$\mathcal{U}$". Let $j: \mathcal{U}' \hookrightarrow \mathcal{U}$. Have compatibilities!

And, $D(\mathrm{Bun}_G) = \lim\limits_{(-)^*} D(\mathcal{U})$

$\Rightarrow$ Get $\mathrm{Loc}: \mathcal{L}\mathfrak{g}\text{-mod}^{\mathcal{L}^+G} \longrightarrow D(\mathrm{Bun}_G)$

But, for Γ, we get

$$D(\mathrm{Bun}_G)_{\mathrm{co}} := \mathrm{colim}\, D(\mathcal{U}) \xrightarrow{\Gamma_{\mathrm{co}}} \mathcal{L}\mathfrak{g}\text{-mod}^{\mathcal{L}^+G}$$

$$D(\mathrm{Bun}_G)^\vee = \Big(\lim_{(-)^*} D(\mathcal{U})\Big)^\vee = \underset{(-)_!\ \vee}{\mathrm{colim}}\ D(\mathcal{U})^\vee = \underset{(-)}{\mathrm{colim}}\ D(\mathcal{U})$$

Break

Tate vector spaces:

$$k((t)) = \mathrm{colim}\,(\cdots \hookrightarrow t\,k[[t]] \hookrightarrow k[[t]] \hookrightarrow \tfrac{1}{t}k[[t]] \hookrightarrow \cdots)$$

$$k[[t]] = \lim\,(\cdots\ k[[t]]/t^2 \longrightarrow k[[t]]/t \to \cdots)$$

V lattice if it's a vector space viewed as union of f.d vector space eg $k((t))/k[[t]]$

V colattice if it's an inverse limit of f.dim vect. spaces. e.g $k[[t]]$.

V fin. dim'l vect. space, $\det(V)$ = graded line, $\mathbb{G}_m$-torsor.

V Tate vect. space, a <u>determinant thy on V</u> is:

- For every colattice $W \subseteq V$, a line $\det(W)$
- For $W \subseteq W'$, $\det(W') \simeq \det(W'/W) \otimes \det(W)$

Collection of determinant theories $:= \delta(V)$, has action of $B\mathbb{G}_m$ since can tensor det. theories w/ any line L.

$\Rightarrow \delta(V)$ is a gerbe $= B\mathbb{G}_m$-torsor.

$\mathcal{L}G \curvearrowright \mathcal{L}\mathfrak{g}$ adjoint $\rightsquigarrow \mathcal{L}G \curvearrowright \delta(\mathcal{L}\mathfrak{g})$, which is the same as a hom. $\mathcal{L}G \longrightarrow B\mathbb{G}_m$. Called the central extension.

Get
$$\begin{tikzcd} \widetilde{\mathcal{L}G} \arrow[r] \arrow[d] & pt \arrow[d] \\ \mathcal{L}G \arrow[r] & B\mathbb{G}_m \end{tikzcd}$$

Tate $= -2\kappa_{crit}$

$\mathcal{L}G_{Tate} = \mathcal{L}\mathfrak{g}_{Tate}$.

<u>Ideology</u>	<u>Set theory</u>	<u>Geometry</u>	
$\mathbb{F}_p$	finite sets	fin. type schemes	G
$\mathbb{Z}_p$	pro-finite sets	infinite type schemes	$\mathcal{L}^+G$
$\mathbb{Q}_p/\mathbb{Z}_p$	infinite sets	ind-schemes of infinite type	Gr_G

locally pro-finite ind-schemes $\mathcal{L}G$

For any $\mathcal{Y}$, there will be:

$$\mathrm{ICoh}_*(\mathcal{Y}) \qquad \mathrm{ICoh}^!(\mathcal{Y})$$

$$f_* \qquad f^!$$

measures functions

In general, are dual (Replace k by Vect)

Step I Inf. dim schemes $\mathcal{L}^+G := \lim \mathcal{L}^+_{\leq n}G$, along faithfully flat maps.

$$\mathcal{L}^+_{\leq n}G = \mathrm{Maps}(\mathrm{Spec}\, k[t]/t^n, G)$$

"a paisant". Let $S = \lim S_\alpha$, S_α finite-type.

$$\mathrm{ICoh}_*(S) = \lim_{(-)_*} \mathrm{ICoh}(S_\alpha) \simeq \mathrm{Colim}_{(-)^*} \mathrm{ICoh}(S_\alpha)$$

$$\mathrm{ICoh}^!(S) = \mathrm{colim}_{(-)^!} \mathrm{ICoh}(S_\alpha)$$

E.g. ω_S = dualizing sheaf $\in \mathrm{ICoh}^!(S)$

$$\mathcal{O}_S \in \mathrm{ICoh}_*(S)$$

Step II apaisant ind-scheme

$$\begin{array}{c} \mathcal{L}G \xrightarrow{\mathcal{L}^+G\text{-torsor}} \mathrm{Gr}_G = \mathrm{colim}\, \mathrm{Gr}_G^{\leq \mu} \\ \text{IS} \\ \mathrm{colim}\, \mathcal{L}^{\leq \mu}G \end{array}$$

Let $\mathcal{Y} = \mathrm{colim}\, \mathcal{Y}_\alpha$, $\mathcal{Y}_\alpha \hookrightarrow \mathcal{Y}_\beta$ almost finite presentation.
be arbitrary $\mathcal{Y}_\alpha$ apaisant scheme.

$$\mathrm{ICoh}_*(\mathcal{Y}) = \operatorname*{colim}_{(-)_*} \mathrm{ICoh}_*(\mathcal{Y}_\alpha)$$ has no canonical element.

$$\mathrm{ICoh}^!(\mathcal{Y}) = \lim_{(-)^!} \mathrm{ICoh}^!(\mathcal{Y}_\alpha) \ni \omega_{\mathcal{Y}}.$$

Step III Equivariance for group-action (extended to stacks)

Ref Raskin, Chen-Fu, but theory not fully-developed yet.

We skip.

To what extent is $\mathrm{ICoh}_* \neq \mathrm{ICoh}^!$?

$$\begin{tikzcd}
X \arrow[d, dotted] & \mathrm{ICoh}_*(X) & \mathrm{ICoh}^!(X) \\
 & \arrow[u] & \arrow[u] \\
X_2 \arrow[d, "f"] & \mathrm{ICoh}(X_2) \arrow[u] \arrow[r, "\otimes \omega_{X_2}"] & \mathrm{ICoh}(X_2) \arrow[u] \\
X_1 \arrow[d] & \mathrm{ICoh}(X_1) \arrow[u, "f^*"] \arrow[r, "\otimes \omega_{X_1}"] & \mathrm{ICoh}(X_1) \arrow[u, "f^!"] \\
X_0 & \arrow[u] & \arrow[u]
\end{tikzcd}$$

For X an apaisant scheme, get canonical equivalence

$$\mathrm{ICoh}_*(X) \longrightarrow \mathrm{ICoh}^!(X)$$
$$\mathcal{O}_X \longmapsto \omega_X.$$

V = Tate vector space viewed as apaisant ind-scheme.

$$V = \operatorname{colim}(\cdots \hookrightarrow V_{-1} \hookrightarrow V_0 \hookrightarrow V_1 \cdots)$$

V_α = apaisant scheme.

$$\mathrm{ICoh}_*(V) \quad \lim\left(\cdots \leftarrow \mathrm{ICoh}_*(V_\alpha) \xleftarrow{i^!} \mathrm{ICoh}_*(V_\beta)\right)$$

$$\mathrm{ICoh}^!(V) : \lim\left(\cdots \leftarrow \mathrm{ICoh}^!(V_\alpha) \xleftarrow{i^!} \mathrm{ICoh}^!(V_\beta) \leftarrow \cdots\right)$$

$$V_\alpha \xhookrightarrow{i} V_\beta$$

Fact: $i : pt \hookrightarrow V \rightsquigarrow i^!(\mathcal{O}_V) = \det(V)$ since

$$\omega_V = \mathcal{O}_V \otimes \det(V)^\vee.$$

Thus,

$$\begin{tikzcd} \mathcal{O}_{V_\alpha} \otimes \det(V_\beta/V_\alpha) \arrow[d, mapsto] & \mathcal{O}_{V_\beta} \arrow[l, mapsto] \arrow[d, mapsto] \\ L_\alpha \otimes \det(V_\beta/V_\alpha) \otimes \omega_{V_\alpha} = L_\beta \otimes \omega_{V_\alpha} & L_\beta \otimes \omega_{V_\beta} \arrow[l, mapsto] \end{tikzcd}$$

So, we get $\mathrm{ICoh}_*(V) \simeq \mathrm{ICoh}^!(V) \otimes \delta(V)$.

$$\omega_X = \det(T_X^\vee),$$

$$\mathrm{Tate}_x = \delta(T_x V), \quad \mathrm{ICoh}_*(\mathcal{Y}) = \mathrm{ICoh}^!_{\mathrm{Tate}}(\mathcal{Y}).$$

Now, let $\mathcal{Y} = B\mathcal{L}G$

$$T\mathcal{Y} = \mathcal{L}\mathfrak{g}[1] \text{ w/ adjoint action of } \mathcal{L}G.$$

shift by 1

? $T^*\mathcal{Y}$ dually $= \delta(\mathcal{L}\mathfrak{g})^{-1}$ w/ induced action

$B\mathcal{L}G^\wedge_{\mathcal{L}^+G}$, $\quad \mathrm{ICoh}_*(B\mathcal{L}G^\wedge_{\mathcal{L}^+G}) = \mathrm{ICoh}^!_{\mathrm{Tate}}(B\mathcal{L}G^\wedge_{\mathcal{L}^+G})$

$$\mathrm{QCoh}_{\alpha}(BG) = \mathrm{QCoh}(B\tilde{G}) \otimes_{\mathrm{QCoh}(B\mathbb{G}_m)} \mathrm{Vect}$$

$$\begin{tikzcd} \tilde{G} \arrow[r] \arrow[d] & pt \arrow[d] \\ G \arrow[r, "\alpha"] & B\mathbb{G}_m \end{tikzcd} \quad \text{homomorphism}$$

$$\begin{tikzcd} B\tilde{G} \arrow[r] \arrow[d] & pt \arrow[d, "\text{universal } B\mathbb{G}_m\text{-torsor}"] \\ BG \arrow[r] & B^2\mathbb{G}_m \end{tikzcd}$$